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Analytic Bethe ansatz solutions for highest states in the su(1|1) and su(2) sectors

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Abstract: We construct the integral equations by taking the thermodynamic limit of both the all-loop Bethe ansatz equation and the string Bethe ansatz equation with the leading strong-coupling dressing factor for the highest states in the $su(1|1)$ and $su(2)$ sectors of the $\mathcal{N}=4$ super Yang-Mills theory. Using the Fourier transformation we solve the integral equations iteratively to obtain the anomalous dimensions of the highest states in the weak coupling expansion.

Keywords: AdS-CFT Correspondence, Bethe Ansatz.

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Contents

1. Introduction

The AdS/CFT correspondence [1] has been deeply revealed by comparing the anomalous dimensions of certain single trace operators in the planar limit of the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the energies of certain string states in the type IIB string theory on $AdS_5 \times S^5$ [2−4]. In particular, integrability has made its appearance in both theories and has shed light on the AdS/CFT correspondence.

The spectrum of anomalous dimension for a local composite operator in the $\mathcal{N} = 4$ SYM theory has been computed by the Bethe ansatz [5] for diagonalization of the dilatation operator [6, 7] that is represented by a Hamiltonian of an integrable spin chain with length L. Further, the asymptotic all-loop Bethe ansatz (BA) equations for the integrable long-range spin chains have been proposed in the $su(2), su(1|1)$ and $sl(2)$ sectors [8−10].

The integrability for the classical $AdS_5 \times S^5$ string sigma model has been investigated by verifying the equivalence between the classical string Bethe equation for the string sigma model and the Bethe equation for the spin chain [11, 12]. Combining the classical string Bethe ansatz and the asymptotic all-loop BA, a set of discrete Bethe ansatz equations for the quantum string sigma model have been constructed [13, 14, 10], where the integrable structure is assumed to be maintained at the quantum level and the quantum string Bethe ansatz (SBA) equations are obtained by modifying the all-loop BA equations with the dressing factor. The SBA has a corresponding perturbative spin chain Hamiltonian at weak coupling so that the SBA equation admits the regular weak coupling expansion [14]. To fix the dressing factor the SBA equation has been studied by comparing its prediction with the quantum world-sheet correction to the spinning string solution [15]. An all-order perturbative expression for the dressing factor at strong coupling has been proposed [16] such that it satisfies the crossing relation [17] and matches with the known physical data at strong coupling [18].

The spectrum of the highest state has been studied by analyzing the all-loop BA equation in the thermodynamic limit $L \to \infty$ for the $su(2)$ sector [19] and the all-loop BA equation and the SBA equation with the leading strong-coupling dressing factor for the $su(1|1)$ sector [20]. The flow of the spectrum from weak to strong coupling has been numerically derived by solving the all-loop BA equation and the SBA equation with the leading strong-coupling dressing factor for the $su(2)$ and $su(1|1)$ sectors in the large but finite L [21]. The strong coupling behavior of the su(2) spectrum has been investigated by using the Hubbard model which is regarded as the microscopic model behind the integrable structure of the $\mathcal{N}=4$ SYM dilatation operator [22]. The highest states for the $su(2)$ and $su(1|1)$ sectors of the $AdS_5 \times S^5$ superstring have been studied analytically in the framework of the light-cone Bethe ansatz equations [23].

For the $sl(2)$ sector the large-spin anomalous dimension of twist-two operator has been computed by solving the all-loop BA equation and the SBA equation with the leading strong-coupling dressing factor in the thermodynamic limit by means of the Fourier transformation [24]. In the former integral equation which we call the ES equation, the anomalous dimension leads to the universal all-loop scaling function $f(g)$ with the gauge coupling constant g satisfying the Kotikov-Lipatov transcendentality [25], whereas in the latter integral equation $f(g)$ is modified at the three-loop order as compared to the ES equation and the transcendentality is not preserved. In the all-loop gauge BA equation with a full weak-coupling dressing factor [26] which is an analytic continuation of a full crossing-symmetric strong-coupling dressing factor [16], which is called the BES equation, the universal scaling function has been shown to be so modified at the four-loop order as to obey the Kotikov-Lipatov transcendentality and be consistent with the planar multi-gluon amplitude of $\mathcal{N}=4$ SYM theory at the four-loop order [27]. The strong coupling behavior of $f(g)$ for the BES equation has been studied numerically by analyzing the equivalent set of linear algebraic equation to reproduce the asymptotic form predicted by the string theory [28]. By truncating the strong coupling expansion of the matrices entering the linear algebraic equation, the strong coupling limit of $f(g)$ has been extracted analytically [29]. In ref. [30] the ES and BES equations have been analyzed by using the Laplace transformation.

Further for the $su(2)$ and $su(1|1)$ sectors the gauge BA equations with the full weak-coupling dressing factor have been analyzed and the anomalous dimensions of the highest states have been presented in the weak coupling expansion [31], where the anomalous dimensions of a state built from a field strength operator and a certain one-loop $so(6)$ singlet state also have been computed. Specially for the $su(2)$ sector a kind of transcendentality has been demonstrated to be preserved. The physical origin of the full weak-coupling dressing factor has been argued [32]. Without resorting to the Fourier transformation the strong coupling solutions for the SBA equations with the leading strong-coupling dressing factor in the rapidity plane have been analytically derived for the highest states in the $su(2)$ and $su(1|1)$ sectors and the strong coupling limit of the universal scaling function $f(g)$ in the $sl(2)$ sector has been estimated from the BES equation by deriving the leading density of Bethe roots in the rapidity plane [33]. On the other hand the Fourier-transform of the SBA equation for the $sl(2)$ sector has been analyzed to study the strong coupling behavior of $f(g)$ [34].

We will analyze the SBA equations with the leading strong-coupling dressing factor for the highest states in the thermodynamic limit $L \to \infty$ for the $su(1|1)$ and $su(2)$ sectors.

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For the $su(2)$ sector we will investigate how the the leading strong-coupling dressing factor produces an effect on the kind of transcendentality which was observed [31] in the gauge BA equation with the full weak-coupling dressing factor. By solving these equations through the Fourier transformation we will derive the anomalous dimensions of the highest states in the weak coupling expansion. Specially the weak coupling spectrum for the $su(1|1)$ sector derived by computing the Fourier-transfomed density of Bethe roots will be compared with the result [20, 21] which was produced by analyzing the SBA equation with the leading strong-coupling dressing factor in the large but finite L and computing the Bethe momenta.

2. Weak coupling spectrum of the highest state in the $su(1|1)$ sector

We consider the highest state in the $su(1|1)$ sector which corresponds to the purely-fermionic operator $\mathrm{tr}(\psi^L)$ [20], where ψ is the highest-weight component of the Weyl spinor from the vector multiplet. The asymptotic all-loop BA equation [10] for the highest state is given by

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{L} \frac{1 - g^2/2x_k^+ x_j^-}{1 - g^2/2x_k^- x_j^+}, \qquad g^2 = \frac{\lambda}{8\pi^2}, \tag{2.1}$$

where u_k $(k = 1, \cdots, L)$ are rapidities of elementary excitations and

$$x_k^\pm = x^\pm(u_k) = \frac{u_\pm}{2}\left(1 + \sqrt{1 - \frac{2g^2}{u_\pm^2}}\right), \qquad u_\pm = u_k \pm \frac{i}{2}. \tag{2.2}$$

The all-loop ansatz (2.1) is a generalization of a three-loop Bethe ansatz [9] and is obtained by deforming the spectral parameter u_k into $x_k^\pm$ in such a way as $u_k \pm i/2 = x_k^\pm + g^2/2x_k^\pm$, where the deformation parameter is the Yang-Mills coupling constant g. The asymptotic all-loop energy $E(g)$ of the highest state is

$$E(g) = g^2 \sum_{k=1}^{L} \left(\frac{i}{x^+(u_k)} - \frac{i}{x^-(u_k)}\right), \tag{2.3}$$

which gives its dimension $\Delta = 3L/2 + E(g)$. Taking the thermodynamic limit in the logarithm of (2.1) and differentiating in the rapidity u we have an integral equation for the density of Bethe roots $\rho(u)$ [20]

$$\frac{1}{i}\left(\frac{1}{\sqrt{u_+^2 - 2g^2}} - \frac{1}{\sqrt{u_-^2 - 2g^2}}\right) = -2\pi\rho(u) - \frac{i}{2}\int_{-\infty}^{\infty} dv\rho(v)\frac{\partial}{\partial u}\log\left(\frac{1 - g^2/2x^+(u)x^-(v)}{1 - g^2/2x^-(u)x^+(v)}\right)^2 \tag{2.4}$$

where $u_\pm = u \pm i/2$ and in the second term the density is integrated against the kernel

$$K_m(u, v) = i\frac{\partial}{\partial u}\log\left(\frac{1 - g^2/2x^+(u)x^-(v)}{1 - g^2/2x^-(u)x^+(v)}\right)^2. \tag{2.5}$$

In this continuum limit the energy shift $E(g)$ (2.3) is also expressed as an integral representation

$$\frac{E(g)}{L} = ig^2 \int_{-\infty}^{\infty} du\rho(u)\left(\frac{1}{x^+(u)} - \frac{1}{x^-(u)}\right). \tag{2.6}$$

Following the Fourier transformation procedure in ref. [24], we solve the integral equation to obtain $E(g)$. The Fourier transform of the density $\rho(u)$ is defined by

$$\hat{\rho}(t) = e^{-|t|/2} \int_{-\infty}^{\infty} du e^{-itu} \rho(u). \tag{2.7}$$

We are interested in the symmetric density $\rho(-u) = \rho(u)$ so that $\hat{\rho}(t)$ is also symmetric $\hat{\rho}(-t) = \hat{\rho}(t)$. Therefore the kernel $K_m(u, v)$ in (2.4) can be symmetrized under the exchange $v \leftrightarrow -v$

$$i\partial_u \log \left(\frac{1-g^2/2x^+(u)x^-(v)}{1-g^2/2x^-(u)x^+(v)} \right)^2 \to \frac{i}{2}\partial_u \log \left(\frac{(1-g^2/2x^+(u)x^-(v))(1+g^2/2x^+(u)x^+(v))}{(1-g^2/2x^-(u)x^+(v))(1+g^2/2x^-(u)x^-(v))} \right)^2, \tag{2.8}$$

which is further described by [24]

$$g^2 \int_{-\infty}^{\infty} dt e^{iut} \int_{-\infty}^{\infty} dt' e^{ivt'} |t| e^{-(|t|+|t'|)/2} \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}g|t'|), \tag{2.9}$$

whose $\hat{K}_m$ is expressed in terms of the Bessel functions as

$$\hat{K}_m(x, x') = \frac{J_1(x)J_0(x') - J_0(x)J_1(x')}{x - x'}. \tag{2.10}$$

We use the expression (2.9) to take the Fourier transformation as $e^{-|t|/2} \int_{-\infty}^{\infty} du e^{-itu} \times$(equation (2.4)) and obtain

$$\hat{\rho}(t) = e^{-|t|} \left(J_0(\sqrt{2}gt) - g^2|t| \int_0^{\infty} dt' \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}(t') \right). \tag{2.11}$$

By solving this integral equation iteratively we derive the transformed density $\hat{\rho}(t)$ expanded in even powers of g as

$$\begin{aligned} \hat{\rho}(t) = e^{-|t|} \Bigg(& 1 - \frac{g^2}{2}(t^2 + |t|) + \frac{g^4}{16}(t^4 + 2(|t|^3 - t^2 + 8|t|)) \\ & - \frac{g^6}{288}(t^6 + 3(|t|^5 - 2t^4 + 26|t|^3 - 60t^2 + 348|t|)) \\ & + \frac{g^8}{9216}(t^8 + 4(|t|^7 - 3t^6 + 54|t|^5 - 246t^4 + 2520|t|^3 - 7200t^2 + 37296|t|)) + \cdots \Bigg). \end{aligned} \tag{2.12}$$

In deriving this solution we have used the following expansion

$$\begin{aligned} \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}gt') = \frac{1}{2}\Bigg(& 1 - \frac{g^2}{4}(t^2 - |t|t' + t'^2) + \frac{g^4}{48}(t^4 - 2|t|^3t' + 4t^2t'^2 - 2|t|t'^3 + t'^4) \\ & - \frac{g^6}{1152}(t^6 - 3|t|^5t' + 9t^4t'^2 - 9|t|^3t'^3 + 9t^2t'^4 - 3|t|t'^5 + t'^6) + \cdots \Bigg). \end{aligned} \tag{2.13}$$

The energy shift $E(g)$ (2.6) can be expressed in terms of the transformed density through (2.7) as

$$\frac{E(g)}{L} = 4g^2 \int_0^{\infty} dt \hat{\rho}(t) \frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt}. \tag{2.14}$$

The substitution of the weak coupling solution (2.12) into (2.14) yields the anomalous dimension of the highest state

$$\frac{E(g)}{L} = 2g^2 - 4g^4 + \frac{29}{2}g^6 - \frac{259}{4}g^8 + \frac{1307}{4}g^{10} + \cdots, \tag{2.15}$$

which reproduces the result of [20, 21].

Now we analyze the SBA equation for the highest state

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^L \frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\sigma^2(x_k,x_j), \tag{2.16}$$

where the leading strong-coupling dressing factor $\sigma(x_k, x_j)$ is defined by

$$\sigma(x_k,x_j) = \left(\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\right)^{-1}\left(\frac{(1-g^2/2x_k^+x_j^-)(1-g^2/2x_k^-x_j^+)}{(1-g^2/2x_k^+x_j^+)(1-g^2/2x_k^-x_j^-)}\right)^{i(u_k-u_j)}. \tag{2.17}$$

In the thermodynamic limit the SBA equation (2.16) becomes an integral equation for the density $\rho(u)$

$$\begin{aligned}\frac{1}{i}\left(\frac{1}{\sqrt{u_+^2-2g^2}} - \frac{1}{\sqrt{u_-^2-2g^2}}\right) &= -2\pi\rho(u) - \frac{1}{2}\int_{-\infty}^{\infty} dv K_m(u,v)\rho(v) \\ &\quad - \int_{-\infty}^{\infty} dv (K_s(u,v) - K_m(u,v))\rho(v),\end{aligned} \tag{2.18}$$

where the main kernel $K_m(u, v)$ is given by (2.5) and

$$K_s(u,v) = -\partial_u(u-v)\log\left(\frac{(1-g^2/2x^+(u)x^-(v))(1-g^2/2x^-(u)x^+(v))}{(1-g^2/2x^+(u)x^+(v))(1-g^2/2x^-(u)x^-(v))}\right)^2. \tag{2.19}$$

The last term of the r.h.s. of (2.18) specifed by $K_s - K_m$ appears as a contribution from the dressing factor, which is compared with (2.4).

In the same way as (2.11) the Fourier transformation of (2.18) leads to

$$\begin{aligned}-e^{-|t|}2\pi J_0(\sqrt{2}gt) &= -2\pi\hat{\rho}(t) + \pi g^2|t|e^{-|t|}\int_{-\infty}^{\infty} dt' \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}g|t'|)\hat{\rho}(t') \\ &\quad -e^{-|t|/2}\int_{-\infty}^{\infty} due^{-itu}\int_{-\infty}^{\infty} dvK_s(u,v)\rho(v).\end{aligned} \tag{2.20}$$

Using the $v \leftrightarrow -v$ symmetrized form of $K_s(u, v)$ we make the third term on the r.h.s. of (2.20) rewritten by [24]

$$-2\pi g^2|t|e^{-|t|}\int_{-\infty}^{\infty} dt'\left(\hat{K}_m(\sqrt{2}g|t|, \sqrt{2}g|t'|) + \sqrt{2}g\tilde{K}(\sqrt{2}g|t|, \sqrt{2}g|t'|)\right)\hat{\rho}(t'), \tag{2.21}$$

where

$$\tilde{K}(x,x') = \frac{x(J_2(x)J_0(x') - J_0(x)J_2(x'))}{x^2 - x'^2}. \tag{2.22}$$

Thus we obtain an integral equation for the transformed density

$$\hat{\rho}(t) = e^{-|t|}\Big(J_0(\sqrt{2}gt) - g^2|t| \int_0^\infty dt' \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}(t') \\ -2g^2|t| \int_0^\infty dt' \sqrt{2}g\tilde{K}(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}(t') \Big). \tag{2.23}$$

Comparing (2.23) with (2.18) and (2.11) we see that the last term in (2.23) specified by $\sqrt{2}g\tilde{K}$ is attributed to the dressing factor so that $\sqrt{2}g\tilde{K}$ is called a dressing kernel.

In order to solve (2.23) by taking the weak coupling expansion we first split the transformed density $\hat{\rho}(t)$ into a main part $\hat{\rho}_0(t)$ and a correction part $\delta\hat{\rho}(t)$ as $\hat{\rho}(t) = \hat{\rho}_0(t)+\delta\hat{\rho}(t)$, where $\hat{\rho}_0(t)$ satisfies the all-loop BA equation (2.11). Therefore we have the following integral equation for $\delta\hat{\rho}(t)$

$$\delta\hat{\rho}(t) = -2g^2|t|e^{-|t|}\Big(\int_0^\infty dt' \sqrt{2}g\tilde{K}(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}_0(t') \tag{2.24}\\ +\frac{1}{2}\int_0^\infty dt' \hat{K}_m(\sqrt{2}g|t|, \sqrt{2}gt')\delta\hat{\rho}_0(t') + \int_0^\infty dt' \sqrt{2}g\tilde{K}(\sqrt{2}g|t|, \sqrt{2}gt')\delta\hat{\rho}_0(t') \Big),$$

where the first term on the r.h.s. is regarded as an inhomogenous one with $\hat{\rho}_0(t')$ already known as (2.12). Using the expansion (2.13) for $\hat{K}_m(x, x')$ and the following weak coupling expansion for $\tilde{K}(x, x')$ with $x = \sqrt{2}g|t|, x' = \sqrt{2}gt'$

$$\tilde{K}(x, x') = \frac{\sqrt{2}g|t|}{8}\left(1 - \frac{g^2}{6}(t^2 + t'^2) + \frac{g^4}{96}(t^4 + 3t^2t'^2 + t'^4) + \cdots \right) \tag{2.25}$$

we determine $\delta\hat{\rho}(t)$ iteratively

$$\delta\hat{\rho}(t) = e^{-|t|}\Big(-\frac{g^4}{2}t^2 + \frac{g^6}{12}(t^4 + 11t^2 + 6|t|) \\ -\frac{g^8}{192}(t^6 + 30t^4 + 24|t|^3 + 384t^2 + 704|t|) + \cdots \Big). \tag{2.26}$$

Combining them we obtain the anomalous dimension $E(g)/L = E_0(g)/L + \delta E(g)/L$ where the main part $E_0(g)/L$ is given by (2.15) and the correction part $\delta E(g)/L$ is evaluated as

$$\frac{\delta E(g)}{L} = -2g^6 + \frac{44}{3}g^8 - \frac{268}{3}g^{10} + \cdots, \tag{2.27}$$

whose expansion starts from the three-loop order. The summation of (2.15) and (2.27) yields the dimension of the highest state

$$\frac{\Delta}{L} = \frac{3}{2} + 2g^2 - 4g^4 + \frac{25}{2}g^6 - \frac{601}{12}g^8 + \frac{2849}{12}g^{10} + \cdots, \tag{2.28}$$

which recovers the result of [20, 21]. Thus we have solved the SBA equation in the thermodynamic limit $L \to \infty$ to derive the Fourier-transformed density iteratively, whereas in [20, 21] the Bethe momenta of excitations in a finite fixed L were iteratively derived.

In [26] the universal scaling function $f(g)$ in the $sl(2)$ sector was obtained from the all-loop gauge BA equation with a weak-coupling dressing factor and $f(g)$ was shown to satisfy the Kotikov-Lipatov transcendentality. Since the dressing factor is universal for the three rank-one sectors, we use it for the $su(1|1)$ sector. In the $sl(2)$ sector, if we compare the integral SBA equation with the leading strong-coupling dressing factor for the transformed density in [24] with the integral gauge BA equation accompanied with the weak-coupling dressing factor in [26], we note that the dressing kernel $\sqrt{2}g\tilde{K}(x,x')$ for the former case corresponds to the dressing kernel $2\hat{K}_c(x,x')$ for the latter case, where $\hat{K}_c(x,x')$ is given by

$$\begin{aligned}
\hat{K}_c(x,x') &= 2g^2\int_0^\infty dt" K_1(x,\sqrt{2}gt")\frac{t"}{e^{t"}-1}K_0(\sqrt{2}gt",x'),\\
K_0(x,x') &= \frac{xJ_1(x)J_0(x')-x'J_0(x)J_1(x')}{x^2-x'^2},\\
K_1(x,x') &= \frac{x'J_1(x)J_0(x')-xJ_0(x)J_1(x')}{x^2-x'^2}.
\end{aligned} \tag{2.29}$$

Therefore by replacing $\sqrt{2}g\tilde{K}(x,x')$ in (2.23) with $2\hat{K}_c(x,x')$ we obtain an integral equation for the transformed density in the $su(1|1)$ sector

$$\begin{aligned}
\hat{\rho}(t) = e^{-|t|}\Bigg(J_0(\sqrt{2}gt) - g^2|t|\int_0^\infty dt'\hat{K}_m(\sqrt{2}g|t|,\sqrt{2}gt')\hat{\rho}(t')\\
-2g^2|t|\int_0^\infty dt' 2\hat{K}_c(\sqrt{2}g|t|,\sqrt{2}gt')\hat{\rho}(t')\Bigg).
\end{aligned} \tag{2.30}$$

Recently this integral equation has been presented and iteratively solved in ref. [31], where the energy modification owing to the weak-coupling dressing factor starts from the four-loop order.

3. Weak coupling spectrum of the highest state in the $su(2)$ sector

We turn to the highest state in the $su(2)$ sector which is described by the antiferromagnetic operator $\mathrm{tr}(Z^{L/2}\Phi^{L/2})+\cdots$ where Z and Φ are charged scalar fields in the $\mathcal{N}=4$ supermultiplet. The asymptotic all-loop BA equation for the highest state is

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{L/2}\frac{x_k^+-x_j^-}{x_k^--x_j^+}\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}, \tag{3.1}$$

whose thermodynamic limit leads to [19]

$$\frac{1}{i}\left(\frac{1}{\sqrt{u_+^2-2g^2}}-\frac{1}{\sqrt{u_-^2-2g^2}}\right) = -2\pi\rho(u) - 2\int_{-\infty}^\infty dv\frac{\rho(v)}{(u-v)^2+1}. \tag{3.2}$$

The Fourier transformation solves the integral equation (3.2) to give an exact expression of the transformed density

$$\hat{\rho}(t) = \frac{J_0(\sqrt{2}gt)}{e^{|t|}+1}, \tag{3.3}$$

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which yields the dimension of the highest state $\Delta = L + E(g)$ in a closed form

$$\frac{E(g)}{L} = 4g^2 \int_0^\infty \frac{dt}{\sqrt{2}gt} \frac{J_0(\sqrt{2}gt)J_1(\sqrt{2}gt)}{e^t+1}. \tag{3.4}$$

We use the following representation of the Riemann zeta function

$$\zeta(n+1) = \frac{1}{(1-2^{-n})n!} \int_0^\infty dt \frac{t^n}{e^t+1} \tag{3.5}$$

to expand (3.4) in g^2

$$\frac{E(g)}{L} = 2\log 2 g^2 - \frac{9}{4}\zeta(3)g^4 + \frac{75}{8}\zeta(5)g^6 - \frac{11025}{256}\zeta(7)g^8 + \frac{112455}{512}\zeta(9)g^{10} + \cdots. \tag{3.6}$$

Let us consider the SBA equation for the highest state

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{L/2} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\sigma^2(x_k,x_j), \tag{3.7}$$

where the leading strong-coupling dressing factor $\sigma(x_k,x_j)$ is given by (2.17). The thermodynamic limit of (3.7) yields an integral equation for the density

$$\begin{aligned}\frac{1}{i}\left(\frac{1}{\sqrt{u_+^2-2g^2}} - \frac{1}{\sqrt{u_-^2-2g^2}}\right) &= -2\pi\rho(u) - 2\int_{-\infty}^{\infty} dv \frac{\rho(v)}{(u-v)^2+1} \\ &\quad - \int_{-\infty}^{\infty} dv (K_s(u,v) - K_m(u,v))\rho(v),\end{aligned} \tag{3.8}$$

where the kernels $K_m(u,v)$ and $K_s(u,v)$ are given by (2.5) and (2.19) respectively. The Fourier transformation of (3.8) through (2.21) gives an integral equation for the transformed density

$$(1+e^{-|t|})\hat{\rho}(t) = e^{-|t|}\left(J_0(\sqrt{2}gt) - 2g^2|t|\int_0^\infty dt' \sqrt{2}g\tilde{K}(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}(t')\right), \tag{3.9}$$

whose last term is the same as the last one in (2.23) for the $su(1|1)$ sector. By using the expansion (2.25) of $\tilde{K}(x,x')$ the transformed density is iteratively solved as $\hat{\rho}(t) = \hat{\rho}_0(t) + \delta\hat{\rho}(t)$ where the main part $\hat{\rho}_0(t)$ is given by (3.3) and the correction part $\delta\hat{\rho}(t)$ has the following weak coupling expansion

$$\begin{aligned}\delta\hat{\rho}(t) = \frac{1}{e^{|t|}+1}\Bigg(&-\frac{g^4}{2}\log 2\, t^2 + \frac{g^6}{12}(\log 2\, t^4 + 6\zeta(3)t^2) \\ &-\frac{g^8}{384}(2\log 2\, t^6 + 33\zeta(3)t^4 + 675\zeta(5)t^2 - 144\log 2\zeta(3)t^2) + \cdots\Bigg).\end{aligned} \tag{3.10}$$

The substitution of (3.3) and (3.10) into (2.14) leads to a separation $E(g)/L = E_0(g)/L + \delta E(g)/L$ where the main part $E_0(g)/L$ takes the expression (3.6) and the correction part $\delta E(g)/L$ is estimated as

$$\begin{aligned}\frac{\delta E(g)}{L} = &-\frac{3}{2}\log 2\zeta(3)g^6 + \left(\frac{75}{8}\log 2\zeta(5) + \frac{3}{2}\zeta(3)^2\right)g^8 \\ &-\left(\frac{6615}{128}\log 2\zeta(7) + \frac{945}{64}\zeta(5)\zeta(3) - \frac{9}{8}\log 2\zeta(3)^2\right)g^{10} + \cdots,\end{aligned} \tag{3.11}$$

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which is compared with the alternating series of (2.27). Further it is noted that the weak coupling expansion of the energy correction induced by the leading strong-coupling dressing factor starts from the three-loop order in the same way as (2.27).

Now for $\sigma(x_k, x_j)$ in (3.7) we use the weak-coupling dressing factor of the BES equation in ref. [26]. From the expression (3.9) we replace the dressing kernel $\sqrt{2}g\tilde{K}(x,x')$ by the dressing kernel $2\hat{K}_c(x,x')$ to obtain

$$(1+e^{-|t|})\hat{\rho}(t) = e^{-|t|}\left(J_0(\sqrt{2}gt) - 2g^2|t|\int_0^\infty dt' 2\hat{K}_c(\sqrt{2}g|t|, \sqrt{2}gt')\hat{\rho}(t')\right), \tag{3.12}$$

whose last term is the same as the last one in (2.30). Recently this integral equation has been derived and solved in ref. [31], where the energy correction also starts from the four-loop order and a kind of transcendentality is observed if a degree of transcendentality is assigned to both the "bosonic" ζ-function (3.5) and the "fermionic" ζ_a-function defined by $\zeta_a(n+1) = (1-2^{-n})\zeta(n+1)$. On the other hand, the summation of (3.6) and (3.11) expressed in terms of $\zeta_a(1) = \log 2$ leads to the following dimension of the highest state

$$\begin{aligned}\frac{\Delta}{L} &= \frac{3}{2} + 2\zeta_a(1)g^2 - \frac{9}{4}\zeta(3)g^4 + \left(\frac{75}{8}\zeta(5) - \frac{3}{2}\zeta_a(1)\zeta(3)\right)g^6 \\ &\quad - \left(\frac{11025}{256}\zeta(7) - \frac{75}{8}\zeta_a(1)\zeta(5) - \frac{3}{2}\zeta(3)^2\right)g^8 \\ &\quad + \left(\frac{112455}{512}\zeta(9) - \frac{6615}{128}\zeta_a(1)\zeta(7) - \frac{945}{64}\zeta(3)\zeta(5) + \frac{9}{8}\zeta_a(1)\zeta(3)^2\right)g^{10} + \cdots,\end{aligned} \tag{3.13}$$

which shows that the kind of transcendentality is not preserved for the SBA equation with the leading strong-coupling dressing factor.

4. Conclusion

We have investigated the SBA equations with the leading strong-coupling dressing factor for the highest states in the $su(1|1)$ and $su(2)$ sectors by applying the Fourier transformation procedure in the rapidity plane and using the expression of the Fourier-transformed dressing kernel. We have computed the anomalous dimensions of the highest states iteratively from the Fourier-transformed SBA equations and presented the alternative derivation of the anomalous dimension in the $su(1|1)$ sector which agrees with the result of [20, 21]. The SBA equation in the thermodynamic limit $L \to \infty$ has been treated and the Fourier-transformed density has been derived iteratively from the integral equation, while in [20, 21] the SBA equation with the leading strong-coupling dressing factor in the large but finite L has been analyzed and the Bethe momenta have been computed iteratively from the SBA equation in the momentum plane.

In the same manner as the SBA equation with the leading strong-coupling dressing factor for the universal scaling function in the $sl(2)$ sector, we have demonstrated that for the SBA equation in the $su(2)$ sector the contribution from the leading strong-coupling dressing factor to the anomalous dimension starts from the three-loop order and there is a violation of the kind of transcendentality presented in ref. [31] for the gauge BA equation with the weak-coupling dressing factor.

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